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(*h*) That the most needed reform in the teaching of mathematics in secondary schools at the present time is an analytical consideration of the subject from the standpoint, first, of its instrumental or vocational use, and, second, its importance as an agency of cultural education. The high school (and equally the college) should then offer as electives quite technical courses in mathematics as an instrumental study, to those desiring the same. These schools should also offer, probably as electives, distinctly cultural courses designed to acquaint young people with the evolution of mathematical studies and the part they now play in the life of the individual and of society, these studies to be much more descriptive and interpretive than anything now found under the heads of algebra and geometry, even including the more historical treatment of these subjects.

DISCUSSIONS.

Relating to the use of i and $e^{\phi i}$ in vector-notation.

(Cf. article "A Note on Plane Kinematics," by Alexander Ziwet and Peter Field, this MONTHLY, Vol. XXI, pp. 105–113.)

DISCUSSION BY EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

In the MONTHLY, April, 1914, Ziwet and Field gave a very interesting and exceedingly neat treatment of certain elementary properties of plane kinematics based on the notation of Burali-Forti and Marcolongo, and they interpreted the analysis in a series of fundamental constructions. The treatment is elegant not alone by what it does, but particularly by what it leaves unsaid. The fundamental operation is multiplication by i or by $e^{\phi i}$, to turn a vector through the angle $\frac{1}{2}\pi$ or φ respectively. It is not mentioned, except once parenthetically, that, used in this sense, i and $e^{\phi i}$ are very different from the ordinary $i = \sqrt{-1}$ and $e^{\phi i} = \cos\varphi + \sqrt{-1}\sin\varphi$ of algebra. Indeed it is only when these operators are applied successively to the same vector that the analogy with complex numbers is valid. When applied in products of vectors the operators are not at all scalar. This may be seen in the following parallel columns, where I use the notations $\times$ and $\wedge$ for scalar and vector products (instead of the respective $\cdot$ and $\times$ of Gibbs).

i and $e^{\phi i}$ as operators	$i = \sqrt{-1},\quad e^{\phi i} = \cos\varphi + \sqrt{-1}\sin\varphi$
$i\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{a} \times i\boldsymbol{a} = 0$	$i\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{a} \times i\boldsymbol{a} = i\boldsymbol{a}^2$
$i\boldsymbol{a} \wedge \boldsymbol{a} = -\boldsymbol{a} \wedge i\boldsymbol{a} \neq 0$	$i\boldsymbol{a} \wedge \boldsymbol{a} = \boldsymbol{a} \wedge i\boldsymbol{a} = 0$
$i\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{a} \times i\boldsymbol{b}$	$i\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times i\boldsymbol{b}$
$i\boldsymbol{a} \times i\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$	$i\boldsymbol{a} \times i\boldsymbol{b} = -\boldsymbol{a} \times b$
$e^{i\phi}\boldsymbol{a} \times e^{i\phi}\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$	$e^{i\phi}\boldsymbol{a} \times e^{i\phi}\boldsymbol{b} = e^{2i\phi}\boldsymbol{a} \times \boldsymbol{b}$
$\boldsymbol{a} \times e^{i\phi}\boldsymbol{b} = \boldsymbol{b} \times e^{-i\phi}\boldsymbol{a}$	$\boldsymbol{a} \times e^{i\phi}\boldsymbol{b} = \boldsymbol{b} \times e^{i\phi}\boldsymbol{a}$
$e^{i\phi}\boldsymbol{a} \times e^{i\psi}\boldsymbol{a} = \boldsymbol{a}^2 \cos(\psi - \varphi)$	$e^{i\phi}\boldsymbol{a} \times e^{i\psi}\boldsymbol{a} = e^{i(\phi+\psi)}\boldsymbol{a}^2$
$e^{i\phi}\boldsymbol{a} \wedge e^{i\psi}\boldsymbol{a} = \boldsymbol{a}^2 \sin(\psi - \varphi)\boldsymbol{u}$	$e^{i\phi}\boldsymbol{a} \wedge e^{i\psi}\boldsymbol{a} = 0.$

In the last row $\boldsymbol{u}$ is a unit vector perpendicular to the plane we are considering.

Now it seems for me that the simple directness of the Ziwet-Field analysis depends largely on the fact that the authors do not investigate the properties of their operator but operate (with one exception) with the assumed rules and knowledge of ordinary scalar complex numbers—a process which does not lead them into error because they do not need to go far enough to get into radical differences with ordinary algebra. It is, however, quite easy to accomplish all their work without introducing any new operators and without laying oneself open to error by using algebraic notations in non-algebraic senses. If $\boldsymbol{u}$ is a unit vector normal to the plane, an operator which turns a vector through a right angle is $\boldsymbol{u}\wedge$. Thus $\boldsymbol{u}\wedge\boldsymbol{a}$ is equal to $\boldsymbol{a}$ in magnitude and perpendicular to $\boldsymbol{a}$ in direction. The operator applied successively to the same vector gives by geometry the ordinary rule for powers of $\sqrt{-1}$; namely,

$$\boldsymbol{u}\wedge(\boldsymbol{u}\wedge\boldsymbol{a}) = -\boldsymbol{a}, \qquad \boldsymbol{u}\wedge(\boldsymbol{u}\wedge(\boldsymbol{u}\wedge\boldsymbol{a})) = -\boldsymbol{u}\wedge\boldsymbol{a}, \quad \cdots.$$

This also follows algebraically by the rule for the reduction of double, triple, $\cdots$, vector products.

We may go further and consider the operator

$$\begin{aligned} e^{\phi \boldsymbol{u}\wedge} &= 1 + \varphi\boldsymbol{u}\wedge + \frac{\varphi^2\boldsymbol{u}\wedge(\boldsymbol{u}\wedge}{2!} + \frac{\varphi^3\boldsymbol{u}\wedge(\boldsymbol{u}\wedge(\boldsymbol{u}\wedge}{3!} + \cdots \\ &= \left(1 - \frac{\varphi^2}{2!} + \cdots\right) + \left(\varphi - \frac{\varphi^3}{3!} + \cdots\right)\boldsymbol{u}\wedge \\ &= \cos\varphi + \sin\varphi\,\boldsymbol{u}\wedge. \end{aligned}$$

We should thus see that we get a formula entirely analogous to that for $e^{\phi i}$; but we should not be tempted to think of the operator as a scalar operator. We should realize at each step that we were dealing with a vector operator. We should, moreover, be right in line with the more general work of A. C. LUNN who, in the MONTHLY, February, 1909, has given a very elegant derivation of the Euler-Rodrigues form for a rotation based upon the operator $\boldsymbol{u}\wedge$ applied without the restriction that the operand should be perpendicular to $\boldsymbol{u}$. Indeed the equations

$$\boldsymbol{u}\wedge(\boldsymbol{u}\wedge\boldsymbol{r}) = (\boldsymbol{u}\times\boldsymbol{r})\boldsymbol{u} - \boldsymbol{r}, \qquad \boldsymbol{u}\wedge(\boldsymbol{u}\wedge(\boldsymbol{u}\wedge\boldsymbol{r})) = -\boldsymbol{u}\wedge\boldsymbol{r}, \quad \cdots$$

show that the general form of the operator is

$$e^{\phi\boldsymbol{u}\wedge} = \cos\varphi + \sin\varphi\,\boldsymbol{u}\wedge + (1 - \cos\varphi)\boldsymbol{u}(\boldsymbol{u}\times.$$

Although I admit the neatness of the i and $e^{\phi i}$ introduced by Burali-Forti and Marcolongo and adopted by Ziwet and Field, I do feel that the elegance thus obtained does not more than make up for the dangers accompanying the notations, to say nothing of the restriction of the operand to a plane and the suppression from the analysis of the vital vector $\boldsymbol{u}$. Is it not likely that a large amount of the appeal of i and $e^{\phi i}$ in this sense is due to our neglect of the operational calculus?

Those who have a little knowledge of this calculus, as applied in solving linear differential equations with constant coefficients, in determining the Bernoulli numbers, and the like,[1] should welcome a chance to reveal, rather than to obscure, that method in the present problem. And to anyone who writes Taylor's expansion for $f(x)$ as $e^{hD}f(x)$, the significance of $e^{\phi \boldsymbol{u} \wedge}$ is apparent.

There is another objection to the specialized non-algebraic use of i and $e^{\phi i}$ as operators and this lies in the fact that their ordinary algebraic use is of importance in the theory of complex vectors.[2] A complex vector or bivector, as Gibbs called it, is written $\boldsymbol{A} + i\boldsymbol{B}$, where $i = \sqrt{-1}$, and is represented by the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ taken together. The expression $e^{\phi i}(\boldsymbol{A} + i\boldsymbol{B})$ treated algebraically gives

$$e^{\phi i}(\boldsymbol{A} + i\boldsymbol{B}) = (\boldsymbol{A} \cos \varphi - \boldsymbol{B} \sin \varphi) + i(\boldsymbol{A} \sin \varphi + \boldsymbol{B} \cos \varphi).$$

If we consider $\boldsymbol{A}$ and $\boldsymbol{B}$ drawn from the same origin, the vectors

$$\boldsymbol{A}' = \boldsymbol{A} \cos \varphi - \boldsymbol{B} \sin \varphi, \qquad \boldsymbol{B}' = \boldsymbol{A} \sin \varphi + \boldsymbol{B} \cos \varphi,$$

drawn from the same origin, terminate on the ellipse constructed with $\boldsymbol{A}$ and $\boldsymbol{B}$ as conjugate radii, and in particular, if we consider rotation from $\boldsymbol{A}$ to $\boldsymbol{B}$ positive, the vectors $\boldsymbol{A}'$ and $\boldsymbol{B}'$ are set back relative to $\boldsymbol{A}$ and $\boldsymbol{B}$ by the (eccentric) angle φ. Thus multiplication by $e^{\phi i}$ has a simple geometric meaning.

Particularly simple are the circular bivectors, namely, bivectors in which $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular and of equal magnitude. Multiplication by $e^{\phi i}$ is in this case ordinary rotation through the angle $-\varphi$. Following the method used by Gibbs in optics we may represent relative motion in a plane by a circular bivector formed by adding to the position vector $\boldsymbol{p} = P - 0$, the velocity $\dot{\boldsymbol{p}}$ multiplied by $i = \sqrt{-1}$ and divided by the angular velocity w. We have

$$\boldsymbol{M} = \boldsymbol{p} + \frac{i}{w}\dot{\boldsymbol{p}}.$$

The motion of all other points in the plane may then be expressed as

$$\boldsymbol{M}' = re^{-\phi i}\boldsymbol{M},$$

where r and φ are polar coördinates of any point with O as pole, $\boldsymbol{p}$ as polar axis and mod $\boldsymbol{p}$ as unit of distance. We shall not try to reproduce the work of Ziwet and Field, but shall content ourselves with having stated that the ordinary $i = \sqrt{-1}$ is interesting and useful in vector analysis, especially in applications to optics.

Note.—It may be worth while, in connection with this Discussion, to add a reference to A. Macfarlane's paper on "A System of Notation for Vector-Analysis; with a Discussion of the Underlying Principles," published by The International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics (Press of The New Era Printing Co., June,

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[1] See WILSON'S *Advanced Calculus*, pp. 149–152, 214–224, 275, 447–449, and passages in Heaviside's *Electromagnetic Theory*.

[2] GIBBS-WILSON, *Vector Analysis*, pp. 426ff.

1912), and to recall that the first two of the fifteen conditions mentioned by him as desirable in a systematic and adequate vector-notation were (in part):

1. It should disturb as little as possible the established notation of mathematical analysis. For example, i should be left to denote $\sqrt{-1}\cdots$.

2. It may generalize, but not contradict, the notation and principles of algebra.—EDITOR.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

TWO NEW TRIGONOMETRIES.

Plane Trigonometry and Tables. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Ginn and Company, Boston, 1914. iv+188 and v+104 pages.

This text is a revised version of the old and widely known book of similar content of the "Wentworth series." With a few exceptions it consists of a partial rearrangement of a portion of the material of the older book, and users of the older book will have no difficulty in recognizing the text and many of the problems of the newer book. Besides some new problems, there have been added a few notes of a historical character, a chapter on logarithms and a chapter on graphs. In this latter chapter the graphs of some of the simpler algebraic functions as well as those of the six trigonometric functions are given. Why does not some author accidentally introduce a new idea into trigonometry by giving the figure for $y = \sin 2x$ or $y = \sin x + \cos x$? The broad treatment of trigonometric equations and identities, and the introduction of the radian are deferred to the end of the text. In its new dress the text will probably lose no old friends and may gain some new ones.

Plane Trigonometry with Tables. By C. I. PALMER and C. W. LEIGH. The McGraw-Hill Book Company, New York, 1914. x+156+132 pages.

The authors make no claim for novelty, new material, or any innovation in this book. They hope to give a better, clearer arrangement of the time-worn subject matter. It is noticeable that the explanation of logarithms forms a separate chapter, bound with the tables, and that in the trigonometry proper there is no crowding of their use into any particular place. The type chosen for the tables seems unfortunate, being tiresome to the eyes if the tables are used for any length of time.

The introductory chapter contains adequate and particularly clear explanations of directed angles and lines together with the graphical addition of each. With this as a basis and with the use of the ideas of rectangular and polar coördinates, the trigonometric functions of the general angle are defined and the simpler developments presented. After this has been done successfully, the special definitions of the trigonometric functions for the right-angled triangle are dragged in from nowhere in particular and for no particularly good reason, and then they are used a great deal. The simpler identities and the inverse functions, involving acute angles, are then introduced.